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Remarks on central extensions of the Galilei group in $2+1$ dimensions

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Abstract

Some properties of central extensions of $2+1$ dimensional Galilei group are discussed. It is shown that certain families of extensions are isomorfic. An interpretation of new nontrivial cocycle is offered. Few bibliographical remarks are included.

1 Introduction

Some attention has been recently paid to the nonrelativistic symmetry of $2+1$–dimensional space–time. The relevant Galilei group differs significantly from its fourdimensiomal counterpart which makes the study of its mathematical structure quite interesting. Moreover, one can expect that such an analysis will appear helpful in understanding the propeties of nonrelativistic systems that are effectivly confined to two spatial dimensions. In papers [1], [2] Bose considers the problem of finding all central extensions of $2+1$–dimensional Galilei group (and its Lie algebra) and constructs the relevant umitary projective representations. However, we feel that not all interesting points were exhausted there. It is the aim of the present short note to add some futher remarks concerning the central extensions of Galilei group / algebra in three dimensions. In section II we prove (actualey, the proof is almost trivial) two theorems indicating that certain families of such extensions are isomorphic and indicate how they can be used to find the relevant Casimir operators and to simplify slighty the representation theory. In section III the explanation is offered for the existence of additional nontrivial cocycle in three dimensions. Namely, it is shown that occurence of this co-

cycle is related to the Thomas precession phenomenon for threedimensional Lorentz group. Finally, the telegraphically short section IV contains some bibligraphical notes. This is because we think the proper credit should be given to authors who obtained the results contained in Ref. [1].

2 On central extensions of 2+1-dimensional Galilean algebra and group

Let M, N_i, H and P_i be rotation, boost, time– and space– translation generators, respectively. Bose [1] has proven that the vector space of central extensions of Lie algebra of Galilei group is threedimensional. In the notation adopted above the extended algebra reads

$$\left.\begin{array}{rcl} [H, P_i] & = & 0 \\ [N_i, H] & = & iP_i \\ [P_i, P_j] & = & 0 \\ [N_i, N_j] & = & ik\varepsilon_{ij}\mathbf{1} \\ [M, P_i] & = & i\varepsilon_{ij}P_j \\ [N_i, P_j] & = & im\delta_{ij}\mathbf{1} \\ [M, N_i] & = & i\varepsilon_{ij}N_j \\ [M, H] & = & il\mathbf{1} \end{array}\right\} \tag{1}$$

the extension being parametrized by three real numbers m, k, l; the central element has been denoted by $\mathbf{1}$.

Let us denote the above algebra by g_{kml}. The following suprisingly simple result holds:

Theorem I:

Let $m \neq 0$, l – arbitrary but fixed. Then g_{kml} are isomorphic, as Lie algebras, for all k.

Proof.

Redefine the basis as follows: $X' = X$, $X \neq N_i$, $N_i' = N_i + \frac{k}{2m}\varepsilon_{ij}P_j$ $\square$

Let us point out that such an isomorphism does not necessarily imply physical

equivalence (cf. Ref. [3]).

As an application we list all Casimir operators for arbitrary k, m, l. It reads

(i) $l = 0$, $m \neq 0$, k—arbitrary

$$\begin{aligned} C_1 &= H - \tfrac{1}{2m}\vec{P}^2, \\ C_2 &= M - \tfrac{1}{m}\vec{N} \times \vec{P} - \tfrac{k}{m}H \end{aligned}$$

(ii) l–arbitrary, $m = 0$, $k = 0$

$$\begin{aligned} C_1' &= \vec{P}^2, \\ C_2' &= \vec{N} \times \vec{P} \end{aligned}$$

(iii) l–arbitrary, $m = 0$, $k \neq 0$

$$C_1'' = \vec{P}^2$$

(iv) $l \neq 0$, $m \neq 0$, k–arbitrary – none.

We are not going to give here the detailed proof but rather content ourselves with few remarks. The case (i) is a straightforward consequence of Theorem I and the analogy with fourdimensional case; (ii) and (iii) are easily verified and only (iv) calls for some comments. Let us put $k = 0$ which, by Theorem I, does not restrict the generality. Let C be the central element of g_{0ml}. It

can be written in “normal“ order as

$$C = \sum_{(\lambda),(\mu),\nu,\rho} c_{(\lambda)(\mu)\nu\rho} \prod_{i=1}^{2} N_i^{\lambda_i} \prod_{i=1}^{2} P_i^{\mu_i} H^\nu M^\rho. \tag{2}$$

Let $\rho_{max}(c)$ be the maximal power of M on the right hand side of eq.(2). Assume that $\rho_{max}(c) > 0$; then

$$C' = C + i(l\rho_{max}(c))^{-1} \cdot [C, C_1] \cdot M = C \tag{3}$$

while $\rho_{max}(c') \leq \rho_{max}(c) - 1$; therfore $\rho_{max}(c) = 0$. Now apply the same reasoning with M replaced by H and C_1 replaced by C_2 (with $k = 0$) to conclude that $\nu_{max}(c) = 0$. We continue this argument by taking N_i and P_i instead of C_i to show that $\lambda_{i\,max}(c) = 0$ and $\mu_{i\,max}(c) = 0$.

Let us now consider the central extensions of Galilei group. The algebras g_{km0} can be integrated to yield the central extensions G_{km} of Galilei group G [1] [3]. They can be described as follows. Let $(\tau, \vec{u}, \vec{v}, R)$ be an element of Galilei group with τ, $\vec{u}$, $\vec{v}$, R being time translation, space traslation, boost and rotation, respectively. Then the multiplication rule for G_{km} reads

$$(\zeta, \tau, \vec{u}, \vec{v}, R) * (\zeta', \tau', \vec{u'}, \vec{v'}, R') =$$

$$= (\zeta\zeta'\omega, \tau + \tau', \overrightarrow{Ru'} + \vec{v} \cdot \tau' + \vec{u}, \overrightarrow{Rv'} + \vec{v}, RR') \tag{4}$$

where $\zeta \in \mathbb{C}$, $|\zeta| = 1$ and non trivial cocycle is given by

$$\omega = \exp\left(-im(\frac{\vec{v}^2}{2}\tau' + \vec{v} \cdot \overrightarrow{Ru'}) - \frac{ik}{2}(\vec{v} \times \overrightarrow{Rv'}) \right) \tag{5}$$

We adopted here the results of Ref[3]; the corresponding cocycle differs by coboundary from the one given in Ref[1].

Theorem I has the following counterpart on the group level.

Theorem II.

Let $m \neq 0$; then all groups G_{km} are isomorphic.

Proof.

Make the following change of parameters:

$$u_i \longrightarrow u_i + \frac{k}{2m}\varepsilon_{ij}v_j,$$

the remaining parameters being unaffected. $\Box$

Again this result appears to be quite useful. In Ref.[2] the induced representations of G_{km} have been found following Mackey's method. However, when attempting to apply this method in straightforward way one is faced with the following apparent difficulty: there seems to be no convenient semidirect product structure for G_{km}. This difficulty was overcome in Ref.[2] by considering the extensions of Galilei group G with the help of two central

charges and selecting the appropriate representations. However, in view of our Theorem II it is unnecessary: we can always assume $k = 0$ or $m = 0$ and in both cases the semidirect structure is transparent. For $l \neq 0$, g_{kml} can be integrated to the central extension $\tilde{G}_{kml}$ of the universal covering $\tilde{G}$ of Galilei goup. The relevant group multiplication rule reads

$$(\zeta, \tau, \vec{u}, \vec{v}, \theta) * (\zeta', \tau', \vec{u}', \vec{v}', \theta') =$$

$$= (\zeta\zeta'\tilde{\omega}, \tau + \tau', \overrightarrow{R(\theta)u'} + \vec{v} \cdot \tau' + \vec{u}, \overrightarrow{R(\theta)v'} + \vec{v}, \theta + \theta'); \tag{6}$$

here $\theta \in \mathbb{R}$,

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

and

$$\tilde{\omega} = exp(i\, l\, \theta\, \tau' - i\, m\, (\frac{\vec{v}^2}{2}\tau' + \vec{v} \cdot \overrightarrow{R(\theta)u'}) - \frac{ik}{2}(\vec{v} \times \overrightarrow{R(\theta)v'})) \tag{7}$$

Theorem II applies here as well. Therefore, it seems that only the case $m = 0$, $l \neq 0$, $k \neq 0$ has to be treated in the way indicated in Ref.[2].

3 The origin of cocycles

We would like to understand the origin of nontrivial cocycles on Galilei group G. It is the more interesting that the relativistic counterpart of G – the Poincare group P – does not admit nontrivial cocycles. On the other hand, G can be obtained from P by contraction procedure. It is therefore desirable to offer some interpretation for emergence of such cocycles in nonrelativistic limit. The following general picture can be given [4]. Let $\omega(g, g')$ be any cocycle on P; write

$$\omega(g, g') = \exp i\xi(g, g') \tag{8}$$

Now, $\omega(g, g')$ is necessarily trivial, i.e. there exists a function ζ on P such that

$$\xi(g, g') = (\delta\zeta)(g, g') \equiv \zeta(gg') - \zeta(g) - \zeta(g'). \tag{9}$$

The exponent $\xi(g, g')$ gives rise to a nontrivial cocycle in the nonrelativistic limit $c \to \infty$ provided it survives the contraction while $\zeta(g)$ does not (typically, it diverges as $c \to \infty$). To make this pictures more concrete let us describe in some detail the contraction procedure. First, we write the

element of Poincare group in matrix form

$$\{\Lambda, a\} \rightarrow \left[\begin{array}{c|c} \Lambda^{\mu}_{\nu} & a^{\mu} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \delta^{\mu}_{\nu} & a^{\mu} \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \Lambda^{\mu}_{\nu} & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{10}$$

The Lorentz matrix $[\Lambda^{\mu}_{\nu}]$ is further decomposed into pure boosts and rotations

$$\Lambda = \mathcal{L}(\vec{v}) \cdot \mathcal{R} \tag{11}$$

where

$$\mathcal{L}(\vec{v}) = \left[\begin{array}{c|c} \gamma & \dfrac{\gamma v_k}{c} \\ \hline \dfrac{\gamma v_i}{c} & \delta_{ik} + (\gamma - 1)\dfrac{v_i v_k}{\vec{v}^2} \end{array}\right], \gamma \equiv \left(1 - \frac{\vec{v}^2}{c^2}\right)^{-\frac{1}{2}} \tag{12}$$

$$\mathcal{R} = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & R \end{array}\right], RR^T = R^T R = I. \tag{13}$$

So, finally

$$\left[\begin{array}{c|c} \Lambda & a \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} I & a \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \mathcal{L}(\vec{v}) & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \mathcal{R} & 0 \\ \hline 0 & 1 \end{array}\right] \tag{14}$$

The contraction limit is now performed by multiplying eq.(14) by X from the right and X^{-1} from the left, where

$$X = \left[\begin{array}{c|c} c & 0 \\ \hline 0 & I \end{array}\right], \tag{15}$$

taking the limit $c \to \infty$ and identifying: $a^0 \to c\tau$, $\vec{a} \to \vec{u}$.

Now, one can easily explain the emergence of standard cocycle related to the mass of particle. Take

$$\zeta(\{\Lambda, a\}) = ca^0$$

in eq.(9). Due to the identification $a^0 = c\tau$, ζ diverges as c^2 in the contraction limit. However,

$$\zeta(\{\Lambda, a\}, \{\Lambda', a'\}) = c(\Lambda^0_\mu a'^\mu + a^0) - ca^0 - ca'^0 = c(\Lambda^0_\mu - \delta^0_\mu)a'^\mu =$$

$$= c^2(\gamma - 1)\gamma' + \gamma v_i R_{ik} u'_k \overset{c\to\infty}{\longrightarrow} \frac{\vec{v}^2}{2} \cdot \tau' + \vec{v} \cdot \overrightarrow{Ru'}. \tag{16}$$

This explanation works both for three and four dimensions.

To account for the second cocycle (related to the parameter k) let us note that, in the case of threedimensional space–time the rotation matrix appearing in the decomposition (11)–(13) of the Lorentz matrix is an element of $SO(2)$ and is therefore characterized by one angle θ. We put

$$\zeta(\{\Lambda, a\}) = c^2\theta(\Lambda). \tag{17}$$

Actually, θ is multivalued no P (while singlevalued on $\tilde{P}$) but this plays no role in what follows. Now, from eq.(9) we get

$$\xi(\Lambda, \Lambda') = c^2((\theta(\Lambda \cdot \Lambda') - \theta(\Lambda) - \theta(\Lambda')). \tag{18}$$

But

$$\theta(\Lambda \cdot \Lambda') = \theta(\Lambda) + \theta(\Lambda') + \delta\theta(\Lambda, \Lambda') \tag{19}$$

where $\delta\theta(\Lambda, \Lambda')$ is $0(1/c^2)$ and is related to the so called Thomas precession [5]; its existence reflects the propety that the composition of pure boosts is no longer a pure boost.

It follows from eqs.(17)–(19) that ξ survives the $c \to \infty$ limit while ζ does not. To calculate $\delta\theta$ we write

$$\Lambda \cdot \Lambda' = (\mathcal{L}(\vec{v})\mathcal{R})(\mathcal{L}(\vec{v'})\mathcal{R}') = \mathcal{L}(\vec{v})(\mathcal{R}\mathcal{L}(\vec{v'})\mathcal{R}^{-1})(\mathcal{R}\mathcal{R}') =$$

$$= (\mathcal{L}(\vec{v})\mathcal{L}(\overrightarrow{Rv'}))(\mathcal{R}\mathcal{R}'). \tag{20}$$

The standard calculations (using eqs.(11), (12), (13)) give

$$\mathcal{L}(\vec{v})\mathcal{L}(\overrightarrow{Rv'}) = \mathcal{L}(\overrightarrow{v''})\mathcal{R}(\delta\theta) \tag{21}$$

where the value of $\vec{v''}$ is there irrelevant while, in the limit $c \to \infty$

$$\delta\theta = \frac{\vec{v} \times \overrightarrow{Rv'}}{2c^2}. \tag{22}$$

By comparying eqs.(18), (19) and (22) we get

$$\xi = \frac{\vec{v} \times \overrightarrow{Rv'}}{2}$$

which gives the cocycle found previously.

4 Bibliographical remarks

We would like to conclude with the following bibligraphical remarks. The central extensions of Lie algebra of threedimensional Galilei group were found many years ago by Levy–Leblond [6]. The corresponding cocycles on Galilei group have been constructed by Grigore [7]. In the same paper Grigore has found the unitary projective representations of $2+1$–dimensional Galilei group using the Mackey theory and exploiting the trick (used again in Ref.[2]) consisting in extending of Galilei group with the help of two (three in the case of universal covering) central charges. Grigore gave also a detailed discussion of projective representations of $2+1$–dimensional Poincare group [8].

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